Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Chaos, Solitons and Fractals 20 (2004) 1059–1073

CHAOS
SOLITONS & FRACTALS

www.elsevier.com/locate/chaos

Periodic and solitary-wave solutions of the Degasperis–Procesi equation

V.O. Vakhnenko [a], E.J. Parkes [b,*]

[a] Institute of Geophysics, Ukrainian Academy of Sciences, 03680 Kyiv, Ukraine
[b] Department of Mathematics, University of Strathclyde, Livingstone Tower, Richmond St., Glasgow G1 1XH, UK

Accepted 29 September 2003

Abstract

Travelling-wave solutions of the Degasperis–Procesi equation are investigated. The solutions are characterized by two parameters. For propagation in the positive x-direction, hump-like, inverted loop-like and coshoidal periodic-wave solutions are found; hump-like, inverted loop-like and peakon solitary-wave solutions are obtained as well. For propagation in the negative x-direction, there are solutions which are just the mirror image in the x-axis of the aforementioned solutions. A transformed version of the Degasperis–Procesi equation, which is a generalization of the Vakhnenko equation, is also considered. For propagation in the positive x-direction, hump-like, loop-like, inverted loop-like, bell-like and coshoidal periodic-wave solutions are found; loop-like, inverted loop-like and kink-like solitary-wave solutions are obtained as well. For propagation in the negative x-direction, well-like and inverted coshoidal periodic-wave solutions are found; well-like and inverted peakon solitary-wave solutions are obtained as well. In an appropriate limit, the previously known solutions of the Vakhnenko equation are recovered.

1. Introduction

As discussed in [1], the family of equations

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{1.1}$$

where b is a constant, contains only two integrable equations, namely the dispersionless Camassa–Holm equation for which $b = 2$ [2] and the Degasperis–Procesi equation (DPE) for which $b = 3$ [3]. In this paper we consider the DPE, namely

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx} \tag{1.2}$$

which may also be written in the form

$$(u_t + uu_x)_{xx} = u_t + 4uu_x. \tag{1.3}$$

It has been known for some time that the dispersionless Camassa–Holm equation has a weak solution in the form of a single 'peakon' [2]

$$u(x,t) = v\mathrm{e}^{-|x-vt|}, \tag{1.4}$$

where v is a constant, and an N-peakon solution [4] that is just a superposition of peakons, namely

* Corresponding author. Tel.: +44-141-548-3720; fax: +44-141-552-8657.
E-mail addresses: vakhnenko@bitp.kiev.ua (V.O. Vakhnenko), ejp@maths.strath.ac.uk (E.J. Parkes).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2003.09.043

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\mathrm{e}^{-|x-q_j(t)|}, \tag{1.5}$$

where the $p_j(t)$ and $q_j(t)$ satisfy a certain associated dynamical system. More recently Degasperis, Holm and Hone [1] proved the integrability of the DPE by constructing its Lax pair, and showed that the equation also has single and N-peakon solutions of the form (1.4) and (1.5) respectively; the peakon dynamics were discussed and compared with the analogous results for Camassa–Holm peakons.

The first aim of this paper is to find all the periodic and solitary-wave solutions to the DPE. Naturally, one of the solitary-wave solutions turns out to be the single peakon solution (1.4).

Some years ago we introduced the Vakhnenko equation (VE) [5], namely

$$\frac{\partial}{\partial x}\mathscr{D}u + u = 0, \quad \text{where } \mathscr{D} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{1.6}$$

and subsequently investigated some of its properties [6–10]. Hone and Wang [11] have shown that there is a subtle connection between the Sawada-Kotera hierarchy and the VE, between the DPE and the VE, and between the Lax pairs of the DPE and VE. In particular they noted that the application of the transformations

$$x \to \tilde{\varepsilon}x - \frac{t}{3\tilde{\varepsilon}}, \quad t \to \tilde{\varepsilon}t, \quad u \to u - \frac{1}{3\tilde{\varepsilon}^2} \tag{1.7}$$

to the DPE (1.3), where $\tilde{\varepsilon}$ is a real positive constant, results in

$$((u_t + uu_x)_x + u)_x = \tilde{\varepsilon}^2(u_t + 4uu_x). \tag{1.8}$$

In the limit $\tilde{\varepsilon} \to 0$, (1.8) reduces to the derivative of the VE for which Hone and Wang found a new Lax pair [11].

In this paper we refer to (1.8) as the transformed DPE. The second aim of this paper is to find a subset of possible periodic and solitary-wave solutions of the transformed DPE and to show that, in the limit $\tilde{\varepsilon} \to 0$, the known solutions of the VE as discussed in [5,6] are recovered.

In Section 2 we show that, for travelling-wave solutions, the DPE may be reduced to a first-order ODE involving two arbitrary constants A and B. We show that there are four distinct periodic solutions corresponding to four different ranges of values of A; for a given allowed value of A, B is restricted to a range of values. By using results established in Appendix A we express the periodic solutions in implicit form; these solutions involve elliptic integrals and Jacobian elliptic functions with parameter m, where $0 < m < 1$. We also investigate the limiting form of these solutions when $m = 1$.

In Section 3 we perform the corresponding analysis for the transformed DPE. We consider the case for which the first-order ODE to which the transformed DPE may be reduced involves only a single integration constant B. We find that there are eight distinct solution regimes corresponding to four different ranges of values of $\tilde{\varepsilon}^2$ and to the two possible directions of propagation. In each case B is restricted to a range of values. We show that, when $\tilde{\varepsilon} \to 0$, the periodic and solitary-wave solutions to the VE are recovered. (For convenience the solutions to the VE as obtained in [5,6] are summarized in Appendix B.)

In Section 4 we summarize our results and make an intriguing speculation.

2. Solutions of the DPE

In this section we seek travelling-wave solutions of the DPE (1.3). Note that there are no bound stationary solutions of (1.3) that are in the form $u = u(x)$. That being the case, it is convenient to introduce a new dependent variable z defined by

$$z = (u - v)/|v| \tag{2.1}$$

and to assume that z is an implicit or explicit function of η, where

$$\eta = x - vt - x_0, \tag{2.2}$$

v and x_0 are arbitrary constants, and $v \neq 0$. Then (1.3) becomes

$$(zz_\eta)_{\eta\eta} = (4z + 3c)z_\eta, \quad \text{where } c : \frac{v}{|v|} = \pm 1. \tag{2.3}$$

After two integrations (2.3) gives

$$(zz_\eta)^2 = f(z), \tag{2.4}$$

where

$$f(z) := z^4 + 2cz^3 + Az^2 + B \equiv (z - z_1)(z - z_2)(z_3 - z)(z_4 - z) \tag{2.5}$$

and A and B are real constants. For the solutions that we are seeking, z_1, z_2, z_3 and z_4 are real constants with $z_1 \leqslant z_2 \leqslant z \leqslant z_3 \leqslant z_4$. Eq. (2.4) is of the same form as (A.1) in Appendix A with $\varepsilon = 1$. Hence we can make use of the solutions given in Appendix A but with $\varepsilon = 1$.

Note that (2.3) is invariant under the transformation $z \to -z$, $c \to -c$; this corresponds to the transformation $u \to -u$, $v \to -v$. Here we will seek the family of solutions of (2.3) for which $v > 0$ and so, from here on in this section, we set $c = 1$.

For convenience we define $g(z)$ and $h(z)$ by

$$f(z) = z^2 g(z) + B, \quad \text{where } g(z) := z^2 + 2z + A, \tag{2.6}$$

and

$$f'(z) = 2zh(z), \quad \text{where } h(z) := 2z^2 + 3z + A, \tag{2.7}$$

and define z_L, z_U, B_L and B_U by

$$z_L := -\frac{1}{4}(3 + \sqrt{9 - 8A}), \quad z_U := -\frac{1}{4}(3 - \sqrt{9 - 8A}), \tag{2.8}$$

$$B_L := -z_L^2 g(z_L) = \frac{A^2}{4} - \frac{9A}{8} + \frac{27}{32} + \frac{1}{32}(9 - 8A)\sqrt{9 - 8A}, \tag{2.9}$$

$$B_U := -z_U^2 g(z_U) = \frac{A^2}{4} - \frac{9A}{8} + \frac{27}{32} - \frac{1}{32}(9 - 8A)\sqrt{9 - 8A}; \tag{2.10}$$

z_L and z_U are the roots of $h(z) = 0$.

Provided A is non-zero and is such that $A < 9/8$, $f(z)$ has three distinct stationary points that occur at $z = z_L$, $z = z_U$ and $z = 0$, and comprise two minimums separated by a maximum. In this case (2.4) has periodic and solitary-wave solutions that have different analytical forms depending on the values of A and B as follows:

2.1. $A < 0$

In this case $z_L < 0 < z_U$ with $f(z_L) < f(z_U)$. For each value of A satisfying $A < 0$ there are periodic inverted loop solutions to (2.4) given by (A.5) and (A.7) with $0 < B < B_U$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 1(a) for an example.

$B = B_U$ corresponds to the limit $z_3 = z_4 = z_U$ so that $m = 1$, and then the solution is an inverted loop-like solitary wave given by (A.9) with $z_2 \leqslant z < z_U$ and

$$z_1 = -\frac{1}{4}(1 + \sqrt{9 - 8A}) - \frac{1}{2}\sqrt{1 + \sqrt{9 - 8A}}, \tag{2.11}$$

$$z_2 = -\frac{1}{4}(1 + \sqrt{9 - 8A}) + \frac{1}{2}\sqrt{1 + \sqrt{9 - 8A}}; \tag{2.12}$$

see Fig. 2(a) for an example. Note that $z_2 \to 0$ and $z_U \to 0$ as $A \to 0$. The amplitude $z_U - z_2$ of the solitary wave increases from 0 as $|A|$ increases from 0.

The loop-like nature of the solitary wave is due to the fact that $z = 0$ is in the range $z_2 \leqslant z < z_U$. For small z, (2.4) gives $z_\eta \simeq \pm\sqrt{B_U}/z$ and so $|z_n| \to \infty$ as $z \to 0$. It follows that the solution curve (see Fig. 2(a) for example) is normal to the η axis at the points $(\mp W/2, 0)$, where W is the maximum width of the loop; from (A.9), W is given by

$$W = 4\tanh^{-1}\left(\sqrt{\frac{z_2}{z_1}}\right) - \frac{2z_U}{p}\tanh^{-1}\left(\sqrt{\frac{z_2}{nz_1}}\right). \tag{2.13}$$

W increases from 0 as $|A|$ increases from 0. Near the points $(\mp W/2, 0)$ the loop is approximately parabolic and given by

$$z^2 \simeq 2\sqrt{B_U}\left(\frac{W}{2} \pm \eta\right). \tag{2.14}$$

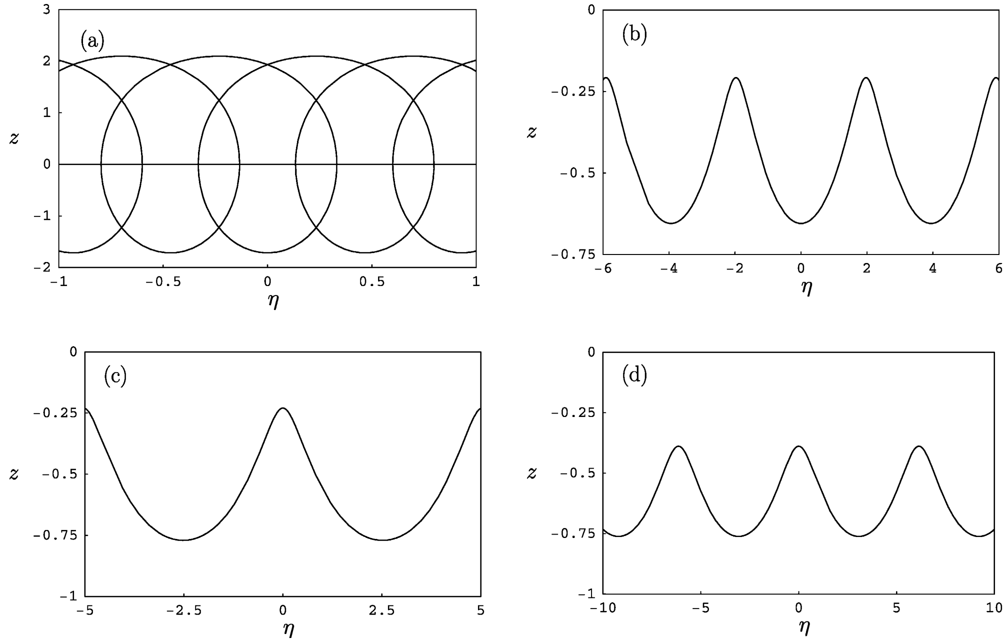

Fig. 1. Periodic solutions of the DPE with $0 < m < 1$: (a) $A = -27$, $B = 0.75B_U$ so $m = 0.842$, $\lambda = 0.466$; (b) $A = 15/16$, $B = 0.5B_U$ so $m = 0.746$, $\lambda = 3.941$; (c) $A = 1$, $B = 0.5B_U = -1/32$ so $m = 0.746$, $\lambda = 5.038$; (d) $A = 135/128$, $B = 0.5(B_U + B_L)$ so $m = 0.729$, $\lambda = 6.140$.

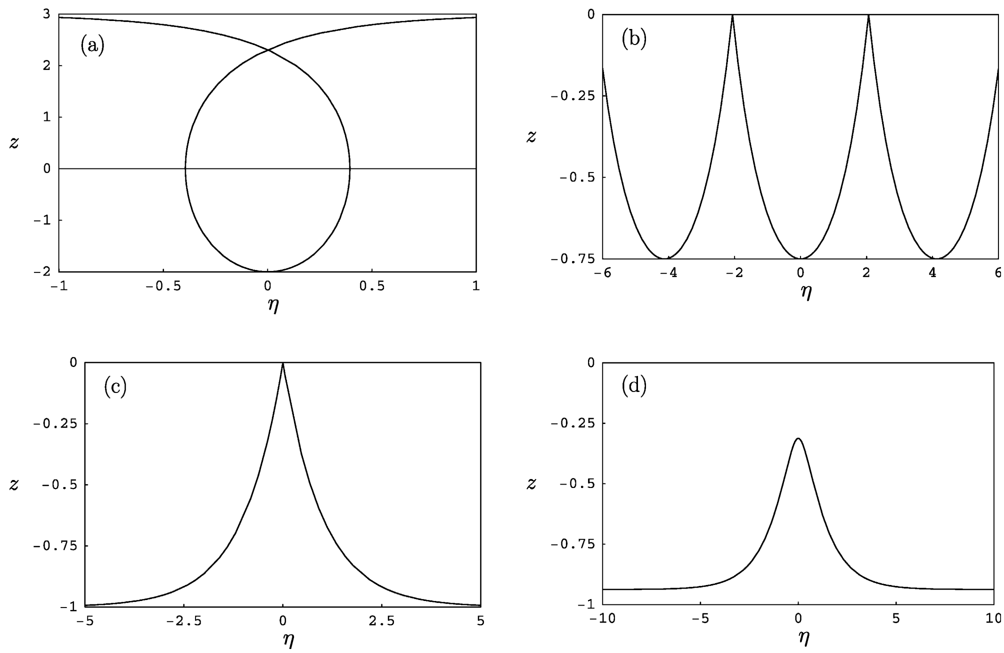

Fig. 2. Solutions of the DPE with $m = 1$: (a) $A = -27$, $B = B_U$, $W = 0.788$; (b) $A = 15/16$, $B = 0$, $\lambda = 4.127$; (c) $A = 1$, $B = 0$; (d) $A = 135/128$, $B = B_L$.

2.2. $0 < A < 1$

In this case $z_L < z_U < 0$ with $f(z_L) < f(0)$. For each value of A satisfying $0 < A < 1$ there are periodic hump solutions to (2.4) given by (A.5) and (A.7) with $B_U < B < 0$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 1(b) for an example.

$B = 0$ corresponds to the limit $z_3 = z_4 = 0$ so that $m = 1$, and then the solution has $z_2 \leqslant z \leqslant 0$ and is given by (A.9) with z_1 and z_2 given by the roots of $g(z) = 0$, namely

$$z_1 = -1 - \sqrt{1-A}, \quad z_2 = -1 + \sqrt{1-A}. \tag{2.15}$$

In this case we obtain a weak solution, namely the periodic upward-cusp wave

$$z = z(\eta - 2j\eta_m), \quad (2j-1)\eta_m \leqslant \eta \leqslant (2j+1)\eta_m, \quad j = 0, \pm 1, \pm 2, \ldots, \tag{2.16}$$

where

$$z(\eta) := [z_2 - z_1 \tanh^2(\eta/2)]\cosh^2(\eta/2) \equiv -1 + \sqrt{1-A}\cosh\eta \tag{2.17}$$

and

$$\eta_m = 2\tanh^{-1}\left(\sqrt{\frac{z_2}{z_1}}\right) \equiv \cosh^{-1}\left(\frac{1}{\sqrt{1-A}}\right); \tag{2.18}$$

see Fig. 2(b) for an example. (2.16) is similar in form to the spatially periodic solution of the Camassa–Holm equation that has been dubbed a 'coshoidal wave' by Boyd [14]. Note that $z_2 \to 0$ and $\eta_m \to 0$ as $A \to 0$, and that $z_2 \to -1$ and $\eta_m \to \infty$ as $A \to 1$. Hence the amplitude $|z_2|$ of the coshoidal wave (2.16) increases from 0 to 1 as A increases from 0 to 1, and its wavelength $\lambda := 2\eta_m$ increases from 0 to infinity.

2.3. $A = 1$

In this case $z_L < z_U < 0$ with $f(z_L) = f(0)$. For $A = 1$ there are periodic hump solutions to (2.4) given by (A.11) and (A.12) with $B_U < B < 0$ so that $0 < m < 1$, where $B_U = -1/16$, and with wavelength given by (A.13); see Fig. 1(c) for an example. An alternative solution is given by (A.5) and (A.7); this is just the former solution phase-shifted by $\lambda/2$.

$B = 0$ corresponds to the limit $z_1 = z_2 = z_L = -1$ and $z_3 = z_4 = 0$. In this case neither (A.9) nor (A.14) is appropriate. Instead we consider (2.4) with $f(z) = z^2(z+1)^2$ and note that the bound solution has $-1 < z \leqslant 0$. On integrating (2.4) and setting $z = 0$ at $\eta = 0$ we obtain the weak solution

$$z = e^{-|\eta|} - 1, \tag{2.19}$$

i.e. a single peakon with amplitude 1; see Fig. 2(c). In terms of the original dependent variable u, (2.19) is equivalent to (1.4) with $v > 0$.

2.4. $1 < A < 9/8$

In this case $z_L < z_U < 0$ with $f(z_L) > f(0)$. For each value of A satisfying $1 < A < 9/8$ there are periodic hump solutions to (2.4) given by (A.11) and (A.12) with $B_U < B < B_L$ so that $0 < m < 1$, and with wavelength given by (A.13); see Fig. 1(d) for an example.

$B = B_L$ corresponds to the limit $z_1 = z_2 = z_L$ so that $m = 1$, and then the solution is a hump-like solitary wave given by (A.14) with $z_L < z \leqslant z_3$ and

$$z_3 = -\frac{1}{4}(1 - \sqrt{9-8A}) - \frac{1}{2}\sqrt{1 - \sqrt{9-8A}}, \tag{2.20}$$

$$z_4 = -\frac{1}{4}(1 - \sqrt{9-8A}) + \frac{1}{2}\sqrt{1 - \sqrt{9-8A}}; \tag{2.21}$$

see Fig. 2(d) for an example. Note that $z_L \to -1$ and $z_3 \to 0$ as $A \to 1$, and that $z_L \to 3/4$ and $z_3 \to -3/4$ as $A \to 9/8$. The amplitude $z_3 - z_L$ of the solitary wave decreases from 1 to 0 as A increases from 1 to 9/8.

3. Solutions of the transformed DPE

In this section we seek travelling-wave solutions of the transformed DPE (1.8). Note that there are no bound stationary solutions of (1.8) that are in the form $u = u(x)$. That being the case, it is convenient to introduce a new dependent variable z defined by

$$z = (u - v)/|v| \tag{3.1}$$

and to assume that z is an implicit or explicit function of η, where

$$\eta = (x - vt - x_0)/|v|^{1/2}, \tag{3.2}$$

v and x_0 are arbitrary constants, and $v \neq 0$. Then, with $\varepsilon = \tilde{\varepsilon}|v|^{1/2}$, (1.8) becomes

$$((zz_\eta)_\eta + z + c)_\eta = \varepsilon^2(4z + 3c)z_\eta, \quad \text{where } c := \frac{v}{|v|} = \pm 1. \tag{3.3}$$

After one integration (3.3) gives

$$(zz_\eta)_\eta + z + c = \varepsilon^2(2z^2 + 3cz) + c_1, \tag{3.4}$$

where c_1 is an arbitrary real constant. Note that, in terms of z and η, the VE (1.6) becomes

$$(zz_\eta)_\eta + z + c = 0 \tag{3.5}$$

(see Eq. (2.2) in [6]). The solitary-wave solution to (3.5) is such that $z_\eta \to 0$, $z_{\eta\eta} \to 0$ and $z + c \to 0$, as $|\eta| \to \infty$. We choose c_1 in (3.4) so that these conditions are satisfied. Accordingly, in this paper we restrict attention to the particular case in which $c_1 = c^2\varepsilon^2$. Then, after one integration, (3.4) gives

$$(zz_\eta)^2 = \varepsilon^2 f(z), \tag{3.6}$$

where

$$f(z) := z^4 - \frac{2}{3\varepsilon^2}(1 - 3c\varepsilon^2)z^3 - \frac{c}{\varepsilon^2}(1 - c\varepsilon^2)z^2 + B \equiv (z - z_1)(z - z_2)(z_3 - z)(z_4 - z) \tag{3.7}$$

and B is a real constant. For the solutions that we are seeking, z_1, z_2, z_3 and z_4 are real constants with $z_1 \leqslant z_2 \leqslant z \leqslant z_3 \leqslant z_4$. Eq. (3.6) is of the same form as (A.1) in Appendix A. Hence we can make use of the solutions given in Appendix A.

For convenience we define $g(z)$ and $h(z)$ by

$$f(z) = z^2 g(z) + B, \quad \text{where } g(z) := z^2 - \frac{2}{3\varepsilon^2}(1 - 3c\varepsilon^2)z - \frac{c}{\varepsilon^2}(1 - c\varepsilon^2), \tag{3.8}$$

and

$$f'(z) = 2zh(z), \quad \text{where } h(z) := 2z^2 - \frac{1}{\varepsilon^2}(1 - 3c\varepsilon^2)z - \frac{c}{\varepsilon^2}(1 - c\varepsilon^2), \tag{3.9}$$

and define z_L, z_U, B_L and B_U by

$$z_L := -c, \quad z_U := \frac{1}{2\varepsilon^2}(1 - c\varepsilon^2), \tag{3.10}$$

$$B_L := -z_L^2 g(z_L) = \frac{c}{3\varepsilon^2}, \tag{3.11}$$

$$B_U := -z_U^2 g(z_U) = \frac{1}{48\varepsilon^8}(1 + 3c\varepsilon^2)(1 - c\varepsilon^2)^3; \tag{3.12}$$

z_L and z_U are the roots of $h(z) = 0$.

Provided that $\varepsilon^2 > 0$ and $\varepsilon^2 \neq 1$, $f(z)$ has three distinct stationary points that occur at $z = z_L$, $z = z_U$ and $z = 0$, and comprise two minimums separated by a maximum. In this case (3.6) has periodic and solitary-wave solutions that have different analytical forms depending on the values of ε^2 and B as follows:

3.1. $c = 1$, $\varepsilon^2 > 1$

In this case $z_L < z_U < 0$ with $f(z_L) < f(0)$. For each value of ε^2 satisfying $\varepsilon^2 > 1$ there are periodic hump solutions to (3.6) given by (A.5) and (A.7) with $B_U < B < 0$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 3(a) for an example.

$B = 0$ corresponds to the limit $z_3 = z_4 = 0$ so that $m = 1$, and then the solution has $z_2 \leqslant z \leqslant 0$ and is given by (A.9) with z_1 and z_2 given by the roots of $g(z) = 0$, where g is defined in (3.8), namely

$$z_1 = \frac{1}{3\varepsilon^2}\left(1 - 3\varepsilon^2 - \sqrt{1 + 3\varepsilon^2}\right), \quad z_2 = \frac{1}{3\varepsilon^2}\left(1 - 3\varepsilon^2 + \sqrt{1 + 3\varepsilon^2}\right). \tag{3.13}$$

In this case we obtain a weak solution, namely the coshoidal wave

$$z = z(\eta - 2j\eta_m), \quad (2j-1)\eta_m \leqslant \eta \leqslant (2j+1)\eta_m, \quad j = 0, \pm 1, \pm 2, \dots, \tag{3.14}$$

where

$$z(\eta) := [z_2 - z_1 \tanh^2(\varepsilon\eta/2)]\cosh^2(\varepsilon\eta/2) \equiv \frac{1}{3\varepsilon^2}\left(1 - 3\varepsilon^2 + \sqrt{1 + 3\varepsilon^2}\cosh(\varepsilon\eta)\right) \tag{3.15}$$

and

$$\eta_m = \frac{2}{\varepsilon}\tanh^{-1}\left(\sqrt{\frac{z_2}{z_1}}\right) \equiv \frac{1}{\varepsilon}\cosh^{-1}\left(\frac{3\varepsilon^2 - 1}{\sqrt{1 + 3\varepsilon^2}}\right); \tag{3.16}$$

see Fig. 4(a) for an example. Note that $z_2 \to 0$ as $\varepsilon^2 \to 1$, and that $z_2 \to -1$ as $\varepsilon^2 \to \infty$. Hence the amplitude $|z_2|$ of the coshoidal wave increases from 0 to 1 as ε^2 increases from 1 to infinity. As ε^2 increases from 1, the wavelength $\lambda := 2\eta_m$ increases from 0, reaches a maximum value of 1.827 at $\varepsilon^2 = 2.769$, and then decreases to 0 as $\varepsilon^2 \to \infty$.

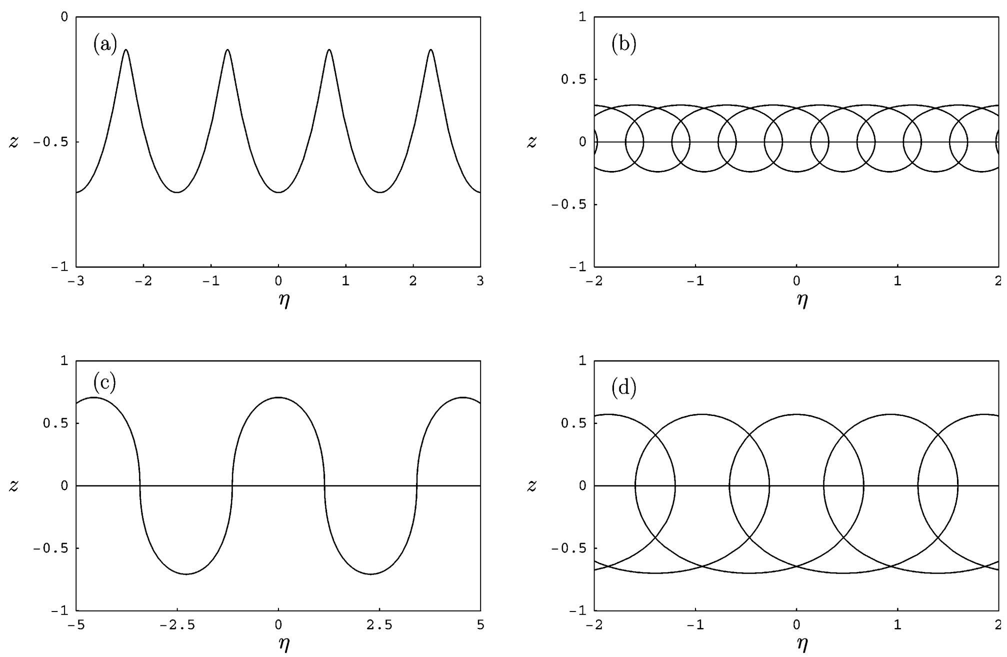

Fig. 3. Periodic solutions of the transformed DPE with $c = 1$ and $0 < m < 1$: (a) $\varepsilon^2 = 8$, $B = 0.25B_U$ so $m = 0.869$, $\lambda = 1.507$; (b) $\varepsilon^2 = 1/2$, $B = 0.6B_U$ so $m = 0.730$, $\lambda = 0.458$; (c) $\varepsilon^2 = 1/3$, $B = 0.75$ so $m = 0.928$, $\lambda = 4.562$; (d) $\varepsilon^2 = 1/4$, $B = 0.75B_L$ so $m = 0.842$, $\lambda = 0.932$.

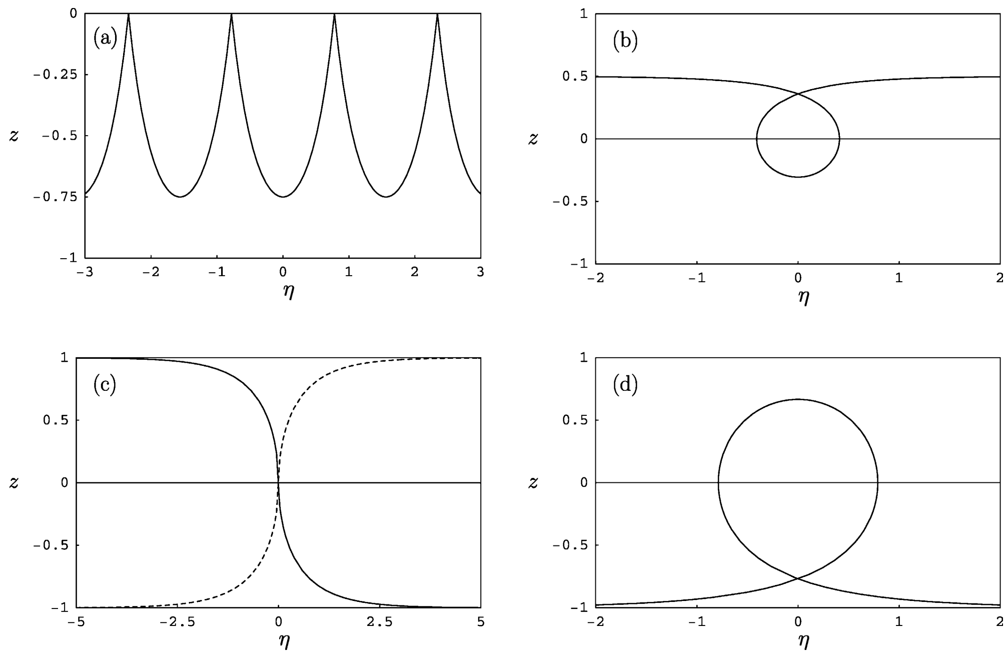

Fig. 4. Solutions of the transformed DPE with $c = 1$ and $m = 1$: (a) $\varepsilon^2 = 8$, $B = 0$, $\lambda = 1.561$; (b) $\varepsilon^2 = 1/2$, $B = B_U$, $W = 0.818$; (c) $\varepsilon^2 = 1/3$, $B = 1$; (d) $\varepsilon^2 = 1/4$, $B = B_L$, $W = 1.577$.

3.2. $c = 1$, $1/3 < \varepsilon^2 < 1$

In this case $z_L < 0 < z_U$ with $f(z_L) < f(z_U)$. For each value of ε^2 satisfying $1/3 < \varepsilon^2 < 1$ there are periodic inverted loop solutions to (3.6) given by (A.5) and (A.7) with $0 < B < B_U$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 3(b) for an example.

$B = B_U$ corresponds to the limit $z_3 = z_4 = z_U$ so that $m = 1$, and then the solution is an inverted loop-like solitary wave given by (A.9) with $z_2 \leqslant z < z_U$ and

$$z_1 = \frac{1}{6\varepsilon^2}\left(-1 - 3\varepsilon^2 - \sqrt{2(9\varepsilon^4 - 1)}\right), \quad z_2 = \frac{1}{6\varepsilon^2}\left(-1 - 3\varepsilon^2 + \sqrt{2(9\varepsilon^4 - 1)}\right); \tag{3.17}$$

see Fig. 4(b) for an example. The maximum width W of the loop is

$$W = \frac{1}{\varepsilon}\left[4\tanh^{-1}\left(\sqrt{\frac{z_2}{z_1}}\right) - \frac{2z_U}{p}\tanh^{-1}\left(\sqrt{\frac{z_2}{nz_1}}\right)\right]. \tag{3.18}$$

Note that $z_2 \to 0$ and $z_U \to 0$ as $\varepsilon^2 \to 1$, and that $z_2 \to -1$ and $z_U \to 1$ as $\varepsilon^2 \to 1/3$. As ε^2 decreases from 1 to 1/3, the amplitude $z_U - z_2$ of the solitary wave increases from 0 to 2, and W increases from 0 to infinity.

3.3. $c = 1$, $\varepsilon^2 = 1/3$

In this case $z_L < 0 < z_U$ with $f(z_L) = f(z_U)$. The z^3 term in the expression for $f(z)$ given by (3.7) is not present and hence $f(z)$ is even so that, for $0 < B < 1$ (with $B_U = B_L = 1$), $z_1 = -z_4$ and $z_2 = -z_3$. Then from the definition of m in (A.6) and the definitions of n in (A.5) or (A.11) we obtain the relation

$$m + n^2 - 2n = 0. \tag{3.19}$$

With (3.19), the results 141.01 and 414.01 in [12] may be used to show that λ given by (A.8) or (A.13) is zero, and hence that η given by (A.7) or (A.12) is periodic in w with period $2K$, where $K := K(m)$ and $K(m)$ is the complete elliptic integral of the first kind. It follows that, for each value of B such that $0 < B < 1$, the solution to (3.6) given by (A.5) and (A.7), or by (A.11) and (A.12), is just a closed curve around the origin in the (η, z) plane. This curve is symmetrical with respect to z and η and has infinite slope at the two points where $z = 0$. A periodic bell solution to (3.6), with wavelength $\lambda := 4\eta(3K/2)$, may be constructed in parametric form as follows:

$$z = z(w), \tag{3.20}$$

$$\eta = \begin{cases} \eta(w) + (2+4j)\eta(3K/2), & -K/2 + 2jK \leqslant w \leqslant K/2 + 2jK, \\ \eta(w) + 4j\eta(3K/2), & K/2 + 2jK \leqslant w \leqslant 3K/2 + 2jK, \end{cases} \tag{3.21}$$

where $z(w)$ and $\eta(w)$ are given by (A.5) and (A.7) respectively, and $j = 0, \pm 1, \pm 2, \ldots$,; see Fig. 3(c) for an example.

$B = B_L = B_U = 1$ corresponds to the limit $z_1 = z_2 = z_L = -1$ and $z_3 = z_4 = z_U = 1$. In this case neither (A.9) nor (A.14) is appropriate. Instead we consider (3.6) with $f(z) = (z+1)^2(1-z)^2$ and note that the bound solutions have $-1 < z < 1$. On integrating (3.6) and setting $z = 0$ at $\eta = 0$ we find that there are two such solutions, namely the kink-like solitary waves

$$z = \begin{cases} +\sqrt{1 - \exp(-2|\eta|/\sqrt{3})}, & \eta < 0, \\ -\sqrt{1 - \exp(-2|\eta|/\sqrt{3})}, & \eta > 0, \end{cases} \tag{3.22}$$

and

$$z = \begin{cases} -\sqrt{1 - \exp(-2|\eta|/\sqrt{3})}, & \eta < 0, \\ +\sqrt{1 - \exp(-2|\eta|/\sqrt{3})}, & \eta > 0; \end{cases} \tag{3.23}$$

see Fig. 4(c) in which the solid and dashed curves correspond to (3.22) and (3.23) respectively.

3.4. $c = 1$, $0 < \varepsilon^2 < 1/3$

In this case $z_L < 0 < z_U$ with $f(z_U) < f(z_L)$. For each value of ε^2 satisfying $0 < \varepsilon^2 < 1/3$ there are periodic loop solutions to (3.6) given by (A.11) and (A.12) with $0 < B < B_L$ so that $0 < m < 1$, and with wavelength given by (A.13); see Fig. 3(d) for an example. For a given choice of B, it is easy to verify numerically that, as ε^2 is made ever smaller (but not zero), the aforementioned solution tends to the solution given by (B.2) with $C = \varepsilon^2 B$; in other words the periodic loop solution of the VE for the case $v > 0$ is recovered in the limit $\varepsilon^2 \to 0$.

$B = B_L$ corresponds to the limit $z_1 = z_2 = z_L = -1$ so that $m = 1$, and then the solution is a loop-like solitary wave given by (A.14) with $-1 < z \leqslant z_3$ and

$$z_3 = \frac{1}{3\varepsilon^2}(1 - \sqrt{1 - 3\varepsilon^2}), \quad z_4 = \frac{1}{3\varepsilon^2}(1 + \sqrt{1 - 3\varepsilon^2}); \tag{3.24}$$

see Fig. 4(d) for an example. The maximum width W of the loop is

$$W = \frac{1}{\varepsilon}\left[4\tanh^{-1}\left(\sqrt{\frac{z_3}{z_4}}\right) - \frac{2}{p}\tanh^{-1}\left(\sqrt{\frac{z_3}{nz_4}}\right)\right]. \tag{3.25}$$

In the limit $\varepsilon^2 \to 0$, it is straightforward to show analytically that the solitary-wave solution reduces to (B.4) and that (3.25) reduces to (B.5); hence, as expected, the loop-like solitary-wave solution of the VE for the case $v > 0$ is recovered.

As ε^2 increases from 0 to 1/3, the amplitude $z_3 + 1$ of the solitary wave increases from 3/2 to 2, and W increases from the value given by (B.5), namely 0.830, to infinity.

3.5. $c = -1$, $0 < \varepsilon^2 < 1/3$

In this case $0 < z_L < z_U$ with $f(0) > f(z_U)$. For each value of ε^2 satisfying $0 < \varepsilon^2 < 1/3$ there are periodic well solutions to (3.6) given by (A.11) and (A.12) with $B_L < B < 0$ so that $0 < m < 1$, and with wavelength given by (A.13); see Fig. 5(a) for an example. For a given choice of B, it is easy to verify numerically that, as ε^2 is made ever smaller (but not zero), the aforementioned solution tends to the solution given by (B.2) with $C = \varepsilon^2 B$; in other words the periodic well solution of the VE for the case $v < 0$ is recovered in the limit $\varepsilon^2 \to 0$.

$B = 0$ corresponds to the limit $z_1 = z_2 = 0$ so that $m = 1$, and then the solution has $0 \leqslant z \leqslant z_3$ and is given by (A.14) with z_3 and z_4 given by the roots of $g(z) = 0$, where g is defined in (3.8), namely

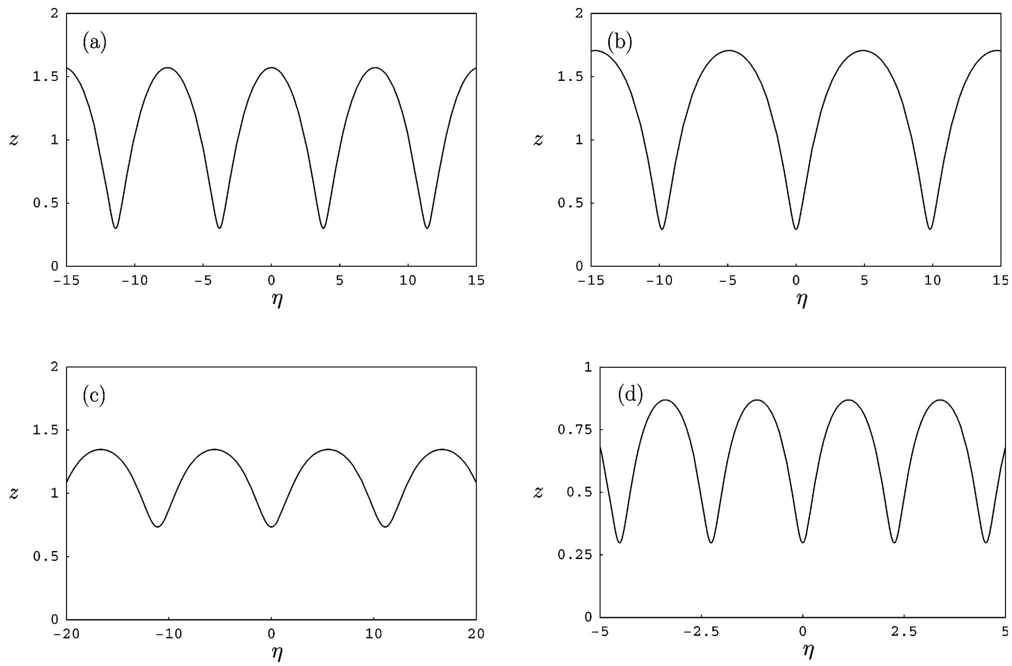

Fig. 5. Periodic solutions of the transformed DPE with $c = -1$ and $0 < m < 1$: (a) $\varepsilon^2 = 1/4$, $B = 0.25B_L$ so $m = 0.842$, $\lambda = 7.600$; (b) $\varepsilon^2 = 1/3$, $B = 0.25B_L = -0.25$ so $m = 0.928$, $\lambda = 9.809$; (c) $\varepsilon^2 = 1/2$, $B = 0.25B_L + 0.75B_U$ so $m = 0.803$, $\lambda = 11.103$; (d) $\varepsilon^2 = 8$, $B = 0.25B_U + 0.75B_L$ so $m = 0.869$, $\lambda = 2.258$.

$$z_3 = \frac{1}{3\varepsilon^2}(1 + 3\varepsilon^2 - \sqrt{1 - 3\varepsilon^2}), \quad z_4 = \frac{1}{3\varepsilon^2}(1 + 3\varepsilon^2 + \sqrt{1 - 3\varepsilon^2}). \tag{3.26}$$

In this case we obtain a weak solution, namely the inverted coshoidal wave

$$z = z(\eta - 2j\eta_m), \quad (2j - 1)\eta_m \leqslant \eta \leqslant (2j + 1)\eta_m, \quad j = 0, \pm 1, \pm 2, \ldots, \tag{3.27}$$

where

$$z(\eta) := [z_3 - z_4 \tanh^2(\varepsilon\eta/2)]\cosh^2(\varepsilon\eta/2) \equiv \frac{1}{3\varepsilon^2}(1 + 3\varepsilon^2 - \sqrt{1 - 3\varepsilon^2}\cosh(\varepsilon\eta)) \tag{3.28}$$

and

$$\eta_m = \frac{2}{\varepsilon}\tanh^{-1}\left(\sqrt{\frac{z_3}{z_4}}\right) \equiv \frac{1}{\varepsilon}\cosh^{-1}\left(\frac{3\varepsilon^2 + 1}{\sqrt{1 - 3\varepsilon^2}}\right); \tag{3.29}$$

see Fig. 6(a) for an example.

In the limit $\varepsilon^2 \to 0$, it is straightforward to show analytically that the inverted coshoidal wave solution (3.27) reduces, as expected, to the inverted paraboidal-wave solution (B.6) of the VE for the case $v < 0$.

As ε^2 increases from 0 to 1/3, the amplitude z_3 of the coshoidal wave increases from 3/2 to 2, and its wavelength $\lambda := 2\eta_m$ increases from 6 to infinity.

3.6. $c = -1$, $\varepsilon^2 = 1/3$

In this case $0 < z_L < z_U$ with $f(0) = f(z_U)$. With $B_L < B < 0$ so that $0 < m < 1$, where $B_L = -1$, there are periodic well solutions to (3.6) given by (A.5) and (A.7), with wavelength given by (A.8); see Fig. 5(b) for an example. An alternative solution is given by (A.11) and (A.12); this is just the former solution phase-shifted by $\lambda/2$.

$B = 0$ corresponds to the limit $z_1 = z_2 = 0$ and $z_3 = z_4 = z_U = 2$. In this case neither (A.9) nor (A.14) is appropriate. Instead we consider (3.6) with $f(z) = z^2(2 - z)^2$ and note that the bound solution has $0 \leqslant z < 2$. On integrating (3.6) and setting $z = 0$ at $\eta = 0$ we obtain the weak solution

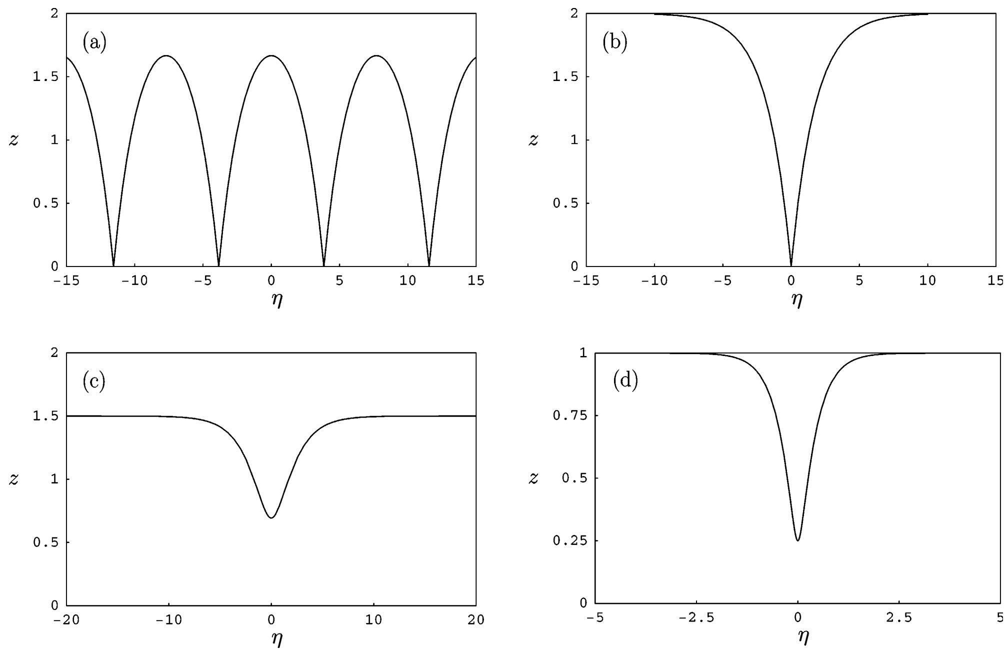

Fig. 6. Solutions of the transformed DPE with $c = -1$ and $m = 1$: (a) $\varepsilon^2 = 1/4$, $B = 0$, $\lambda = 7.699$; (b) $\varepsilon^2 = 1/3$, $B = 0$; (c) $\varepsilon^2 = 1/2$, $B = B_U$; (d) $\varepsilon^2 = 8$, $B = B_L$.

$$z = 2(1 - \exp[-|\eta|/\sqrt{3}]), \tag{3.30}$$

i.e. a single inverted peakon with amplitude 2; see Fig. 6(b).

3.7. $c = -1$, $1/3 < \varepsilon^2 < 1$

In this case $0 < z_L < z_U$ with $f(0) < f(z_U)$. For each value of ε^2 satisfying $1/3 < \varepsilon^2 < 1$ there are periodic well solutions to (3.6) given by (A.5) and (A.7) with $B_L < B < B_U$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 5(c) for an example.

$B = B_U$ corresponds to the limit $z_3 = z_4 = z_U$ so that $m = 1$, and then the solution is a well-like solitary wave given by (A.9) with $z_2 \leqslant z < z_U$ and

$$z_1 = \frac{1}{6\varepsilon^2}(-1 + 3\varepsilon^2 - \sqrt{2(9\varepsilon^4 - 1)}), \quad z_2 = \frac{1}{6\varepsilon^2}(-1 + 3\varepsilon^2 + \sqrt{2(9\varepsilon^4 - 1)}); \tag{3.31}$$

see Fig. 6(c) for an example. Note that $z_2 \to 1$ and $z_U \to 1$ as $\varepsilon^2 \to 1$, and that $z_2 \to 0$ and $z_U \to 2$ as $\varepsilon^2 \to 1/3$. As ε^2 decreases from 1 to 1/3, the amplitude $z_U - z_2$ the solitary wave increases from 0 to 2.

3.8. $c = -1$, $\varepsilon^2 > 1$

In this case $0 < z_U < z_L$ with $f(0) < f(z_L)$. For each value of ε^2 satisfying $\varepsilon^2 > 1$ there are periodic well solutions to (3.6) given by (A.5) and (A.7) with $B_U < B < B_L$ so that $0 < m < 1$, and with wavelength given by (A.8); see Fig. 5(d) for an example.

$B = B_L$ corresponds to the limit $z_3 = z_4 = z_L = 1$ so that $m = 1$, and then the solution is a well-like solitary wave given by (A.9) with $z_2 \leqslant z < z_L$ and

$$z_1 = \frac{1}{3\varepsilon^2}(1 - \sqrt{1 + 3\varepsilon^2}), \quad z_2 = \frac{1}{3\varepsilon^2}(1 + \sqrt{1 + 3\varepsilon^2}); \tag{3.32}$$

see Fig. 6(d) for an example. Note that $z_2 \to 1$ as $\varepsilon^2 \to 1$, and that $z_2 \to 0$ as $\varepsilon^2 \to \infty$. As ε^2 increases from 1 to infinity, the amplitude $1 - z_2$ of the solitary wave increases from 0 to 1.

4. Summary and conclusion

We have found expressions for the travelling-wave solutions to the DPE that travel in the positive x-direction with speed v. These solutions depend, in effect, on two parameters A and m. In addition to the expected single peakon solution (with $A=1, m=1$) there are inverted loop-like ($A<0, m=1$) and hump-like ($1<A<9/8, m=1$) solitary-wave solutions. For $0<m<1$ there are periodic inverted loop ($A<0$) and periodic hump ($0<A<9/8$) solutions. For $m=1$ and $0<A<1$ there are (periodic) coshoidal solutions. For each of the aforementioned solutions expressed with u as the dependent variable, there is a solution for u that is the mirror image in the x-axis and travels with the same speed but in the opposite direction.

We have also found expressions for the travelling-wave solutions to the transformed DPE. These solutions depend, in effect, on two parameters ε^2 and m, and also on the direction of propagation.

For propagation in the positive x-direction there are inverted loop-like ($1/3<\varepsilon^2<1$, $m=1$), kink-like ($\varepsilon^2=1/3$, $m=1$) and loop-like ($0<\varepsilon^2<1/3$, $m=1$) solitary-wave solutions. For $0<m<1$ there are periodic hump ($\varepsilon^2>1$), periodic inverted-loop ($1/3<\varepsilon^2<1$), periodic bell ($\varepsilon^2=1/3$) and periodic loop ($0<\varepsilon^2<1/3$) solutions. For $m=1$ and $\varepsilon^2>1$ there are (periodic) coshoidal solutions. In the limit $\varepsilon^2\to 0$, the periodic loop solutions ($0<m<1$) and loop-like solitary-wave solutions ($m=1$) to the VE are recovered.

For propagation in the negative x-direction there are inverted peakon ($\varepsilon^2=1/3$, $m=1$) and well-like ($1/3<\varepsilon^2<1$ and $\varepsilon^2>1$, $m=1$) solitary-wave solutions. For $0<m<1$ there are periodic well ($0<\varepsilon^2<1$ and $\varepsilon^2>1$) solutions. For $m=1$ and $0<\varepsilon^2<1/3$ there are (periodic) inverted coshoidal solutions. In the limit $\varepsilon^2\to 0$, the periodic well solutions ($0<m<1$) and (periodic) inverted paraboidal solutions ($m=1$) to the VE are recovered.

Recently we have introduced and investigated generalizations of the VE, namely the 'generalized Vakhnenko equation' [15,16]

$$\frac{\partial}{\partial x}\left(\mathscr{D}^2u+\frac{1}{2}u^2+\beta u\right)+\mathscr{D}u=0 \tag{4.1}$$

or equivalently

$$\left(\frac{\partial u}{\partial x}+\mathscr{D}\right)\left(\frac{\partial}{\partial x}\mathscr{D}u+u+\beta\right)=0,$$

where β is an arbitrary real constant, and the 'modified generalized Vakhnenko equation' [17]

$$\frac{\partial}{\partial x}(\mathscr{D}^2u+qu^2+\beta u)+q\mathscr{D}u=0, \tag{4.2}$$

where q is an arbitrary positive constant. These equations have hump-like, loop-like and cusp-like soliton solutions. It is interesting to speculate whether either of these new equations has a counterpart that is related to the DPE or a generalization of it.

Acknowledgements

The contribution of V.O.V. to this research was supported in part by STCU, Project N1747.

Appendix A

Here we consider solutions to

$$(zz_\eta)^2=\varepsilon^2 f(z), \tag{A.1}$$

where

$$f(z):=(z-z_1)(z-z_2)(z_3-z)(z_4-z); \tag{A.2}$$

for the solutions that we are seeking, z_1, z_2, z_3 and z_4 are real constants with $z_1\leqslant z_2\leqslant z\leqslant z_3\leqslant z_4$.

Following [6] we introduce ζ defined by

$$\frac{d\eta}{d\zeta}=\frac{z}{\varepsilon} \tag{A.3}$$

so that (A.1) becomes

$$z_\zeta^3 = f(z). \tag{A.4}$$

(A.4) has two possible forms of solution.

The first form of solution of (A.4) is found using result 254.00 in [12]. It is

$$z = \frac{z_2 - z_1 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)} \quad \text{with } n = \frac{z_3 - z_2}{z_3 - z_1}, \tag{A.5}$$

where

$$w = p\zeta, \quad p = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)} \quad \text{and} \quad m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}. \tag{A.6}$$

In (A.5) $\mathrm{sn}(w|m)$ is a Jacobian elliptic function, where the notation is as used in Chapter 16 of [13]. On using result 400.01 in [12] we find from (A.5) and (A.3) that

$$\eta = \frac{1}{\varepsilon p}[wz_1 + (z_2 - z_1)\Pi(n; w|m)], \tag{A.7}$$

where $\Pi(n; w|m)$ is the elliptic integral of the third kind and the notation is as used in Section 17.2.15 of [13]. The solution to (A.1) is given in parametric form by (A.5) and (A.7) with w as the parameter. With respect to w, z in (A.5) is periodic with period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. It follows from (A.7) that the wavelength λ of the solution to (A.1) is

$$\lambda = \frac{2}{\varepsilon p}|z_1 K(m) + (z_2 - z_1)\Pi(n|m)|, \tag{A.8}$$

where $\Pi(n|m)$ is the complete elliptic integral of the third kind. When $z_3 = z_4$, $m = 1$ and so (A.5) and (A.7) become

$$z = \frac{z_2 - z_1 n \tanh^2 w}{1 - n\tanh^2 w}, \quad \eta = \frac{1}{\varepsilon}\left[\frac{wz_3}{p} - 2\tanh^{-1}(\sqrt{n}\tanh w)\right]. \tag{A.9}$$

In (A.9) η was obtained by using

$$\Pi(n; w|1) = \frac{1}{1-n}\left[w - \sqrt{n}\tanh^{-1}(\sqrt{n}\tanh w)\right], \tag{A.10}$$

cf. result 111.04 in [12].

The second form of solution of (A.4) is found using result 255.00 in [12]. It is

$$z = \frac{z_3 - z_4 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)} \quad \text{with } n = \frac{z_3 - z_2}{z_4 - z_2}, \tag{A.11}$$

where w, p and m are as in (A.6). On using result 400.01 in [12] we find from (A.11) and (A.3) that

$$\eta = \frac{1}{\varepsilon p}[wz_4 - (z_4 - z_3)\Pi(n; w|m)]. \tag{A.12}$$

The solution to (A.1) is given in parametric form by (A.11) and (A.12) with w as the parameter. The wavelength of this solution is

$$\lambda = \frac{2}{\varepsilon p}|z_4 K(m) - (z_4 - z_3)\Pi(n|m)|. \tag{A.13}$$

When $z_1 = z_2$, $m = 1$ and so (A.11) and (A.12) become

$$z = \frac{z_3 - z_4 n \tanh^2 w}{1 - n\tanh^2 w}, \quad \eta = \frac{1}{\varepsilon}\left[\frac{wz_2}{p} + 2\tanh^{-1}(\sqrt{n}\tanh w)\right]. \tag{A.14}$$

Appendix B

Here we summarize the solutions to the VE obtained in [5,6] but in a notation consistent with that used in Section 3.

The VE (3.5) may be integrated once to obtain the equation given by setting $\varepsilon = 0$ in (3.6) and (3.7), namely

$$(zz_\eta)^2 = -\frac{2z^3}{3} - cz^2 + C \equiv (z - z_1)(z - z_2)(z_3 - z), \tag{B.1}$$

where $C(\equiv \varepsilon^2 B)$ is a real constant. The required solutions to (B.1) are such that $z_1 \leqslant z_2 \leqslant z \leqslant z_3$, where z_1, z_2 and z_3 are real constants. These solutions are given implicitly by

$$z = z_3 - (z_3 - z_2)\mathrm{sn}^2(w|m), \quad \eta = \frac{\sqrt{6}}{\sqrt{z_3 - z_1}}[z_1 w + (z_3 - z_1)E(w|m)], \tag{B.2}$$

where $m = (z_3 - z_2)/(z_3 - z_1)$ and $E(w|m)$ is the elliptic integral of the second kind.

The wavelength of the solution (B.2) is

$$\lambda = \frac{2\sqrt{6}}{\sqrt{z_3 - z_1}}|z_1 K(m) + (z_3 - z_1)E(m)|, \tag{B.3}$$

where $E(m)$ is the complete elliptic integral of the second kind.

The VE has two families of solutions corresponding to $v > 0$ and $v < 0$ respectively, where v is as defined in (3.1) and (3.2).

With $v > 0$ we have $c = 1$. Then, with $0 < C < 1/3$, there are periodic loop solutions given by (B.2) with $0 < m < 1$. With $C = 1/3$, $z_1 = z_2 = -1$ and $z_3 = 1/2$ so that $m = 1$ and then (B.2) reduces to the loop-like solitary wave of amplitude 3/2 given by

$$z = \frac{1}{2} - \frac{3}{2}\tanh^2 w, \quad \eta = -2w + 3\tanh w, \tag{B.4}$$

with maximum loop width

$$W = 2\sqrt{3} - 4\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right). \tag{B.5}$$

In [7] we showed that the solitary wave given by (B.4) is a soliton.

With $v < 0$ we have $c = -1$. Then, with $-1/3 < C < 0$, there are periodic well solutions given by (B.2) with $0 < m < 1$. With $C = 0$, $z_1 = z_2 = 0$ and $z_3 = 3/2$ so that $m = 1$ and then (B.2) reduces to the spatially periodic inverted 'paraboidal' wave of amplitude 3/2 given by

$$z = z(\eta - 6j), \quad -3 \leqslant \eta - 6j \leqslant 3, \quad j = 0, \pm 1, \pm 2, \ldots, \tag{B.6}$$

where

$$z(\eta) := \frac{3}{2} - \frac{1}{6}\eta^2. \tag{B.7}$$

References

[1] Degasperis A, Holm DD, Hone ANW. A new integrable equation with peakon solutions. Theor Math Phys 2002;133:1463–74.
[2] Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. Phys Rev Lett 1993;71:1661–4.
[3] Degasperis A, Procesi M. Asymptotic integrability. In: Degasperis A, Gaeta G, editors. Symmetry and perturbation theory. Singapore: World Scientific; 1999. p. 23–37.
[4] Camassa R, Holm DD, Hyman JM. A new integrable shallow water equation. Adv Appl Mech 1994;31:1–33.
[5] Vakhnenko VA. Solitons in a nonlinear model medium. J Phys A Math Gen 1992;25:4181–7.
[6] Parkes EJ. The stability of solutions of Vakhnenko's equation. J Phys A Math Gen 1993;26:6469–75.
[7] Vakhnenko VO, Parkes EJ. The two loop soliton of the Vakhnenko equation. Nonlinearity 1998;11:1457–64.
[8] Morrison AJ, Parkes EJ, Vakhnenko VO. The N loop soliton solution of the Vakhnenko equation. Nonlinearity 1999;12:1427–37.
[9] Vakhnenko VO, Parkes EJ, Michtchenko AV. The Vakhnenko equation from the viewpoint of the inverse scattering method for the KdV equation. Int J Diff Eqns Applicat 2000;1:429–49.

[10] Vakhnenko VO, Parkes EJ. The calculation of multi-soliton solutions of the Vakhnenko equation by the inverse scattering method. Chaos, Solitons & Fractals 2002;13:1819–26.
[11] Hone ANW, Wang JP. Prolongation algebras and Hamiltonian operators for peakon equations. Inverse Probl 2003;19:129–45.
[12] Byrd PF, Friedman MD. Handbook of elliptic integrals for engineers and scientists. Berlin: Springer; 1971.
[13] Abramowitz M, Stegun IA. Handbook of mathematical functions. New York: Dover; 1972.
[14] Boyd JP. Peakons and coshoidal waves: traveling wave solutions of the Camassa–Holm equation. Appl Math Comput 1997;81:173–87.
[15] Morrison AJ, Parkes EJ. The *N*-soliton solution of a generalised Vakhnenko equation. Glasgow Math J 2001;43:65–90.
[16] Vakhnenko VO, Parkes EJ, Morrison AJ. A Bäcklund transformation and the inverse scattering transform method for the generalised Vakhnenko equation. Chaos, Solitons & Fractals 2003;17:683–92.
[17] Morrison AJ, Parkes EJ. The *N*-soliton solution of the modified generalised Vakhnenko equation (a new nonlinear evolution equation). Chaos, Solitons & Fractals 2003;16:13–26.